a review of Higher categories and homotopical algebra by Cisinski, Denis-Charles

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Both category theory and homotopy theory have their origin in algebraic topology. Nevertheless, classical category theory and classical homotopy theory had been set apart in tradition, because homotopical algebra loomed through the dust of peculiarities. Indeed, homotopical algebra is neither complete nor concrete from the standpoint of a category theorist [*P. Freyd*, Lect. Notes Math. 168, 25–34 (1970; Zbl 0212.55801); Repr. Theory Appl. Categ. 2004, No. 6, 1–10 (2004; Zbl 1057.18001)]. The barrier between the two fields started to fade in the late 1990s with the rise of ∞-categories as we know them today owing to the great contributions of André Joyal, Charles Rezk, Bertrand Toën, Gabriele Vezzosi, Carlos Simpson, Jacob Lurie and others. The principal objective in this book is to introduce the basic aspects of the theory of ∞-categories, which enables one to implement the methods of algebraic topology in broader contexts such as algebraic geometry [*B. Toën* and *G. Vezzosi*, Adv. Math. 193, No. 2, 257–372 (2005; Zbl 1120.14012); Homotopical algebraic geometry. II: Geometric stacks and applications. Providence, RI: American Mathematical Society (AMS) (2008; Zbl 1145.14003); *J. Lurie*, "Higher algebra", `http://people.math.harvard.edu/~lurie/papers/HA.pdf`]. It is not only a new approach to the foundations of mathematics but also a so powerful gadget as to bring out many spectacular advances such as the proof of Weil's conjecture on Tamagawa numbers over function fields by Lurie and Gaitsgory and the modern approach to p-adic Hodge theory by Bhatt, Morrow and Scholze [*B. Bhatt* et al., Math. Res. Lett. 22, No. 6, 1601–1612 (2015; Zbl 1349.14070); Publ. Math., Inst. Hautes Étud. Sci. 129, 199–310 (2019; Zbl 07059677); Publ. Math., Inst. Hautes Étud. Sci. 128, 219–397 (2018; Zbl 07018374)]. The intertwining thesis of the book is that the theory of ∞-categories is a semantic interpretation of the formal language of category theory. Model categories are eventually allowed to be ∞-categories themselves and it is shown that the localization of a model category is also a model category, as far as the weak equivalences are invertible maps and the fibrations are all maps.

Chapter 2 aims at introducing fundamentals of Quillen's theory of model category structures needed to manipulate ∞-categories, beginning with a recollection on factorisation systems. The second half of the chapter proceeds on the lines of [*D.-C. Cisinski*, Les préfaisceaux comme modèles des types d'homotopie. Paris: Société Mathématique de France (2006; Zbl 1111.18008)], introducing a method to construct model category structures from scratch on categories of presheaves in case that the cofibrations are defined to be monomorphisms.

Chapter 3 is concerned with the homotopy theory of ∞-categories. The first section constructs the classical Kan-Quillen model category structure on the category of simplicial sets. Using Kan's subdivision functor through classical arguments on diagonals of bisimplicial sets, it is shown that the fibrations are precisely the Kan fibrations. The second section, being technical but fundamental, addresses the compatibility of the homotopy theory of ∞-categories with finite Cartesian products. The third section defines the Joyal model category structure. The fourth section, taken entirely from Joyal's work, introduces fundamental constructions such as joins and slices, as well as a non-trivial lifting property which expresses the fact that, although one can not compose maps canonically, one can choose inverses coherently in ∞-categories. It is exploited in the fifth section first to establish that the Kan complexes are exactly the ∞-groupoids and secondly to prove that a natural transformation is invertible iff it is fiberwise invertiable. After revisiting features of the Joyal model category structures for a couple of sections, the eithth section returns to classical homotopy theory, discussing the Serre long exact sequence associated to a Kan fibration, from which the simplicial version of Whitehead's theorem is derived. This is used in the last section to establish a kind of generalization of ∞-categories claiming that a functor between ∞-categories is an equivalence of ∞-categories iff it is fully faithful and essentaily surjective.

The first paragraph of the first section of Chapter 4 is heuristic, explaining why it is natural to put right fibrations $X \to C$ down as presheaves over C. The rest of the section is concerned with the construction of the homotopy theory of right fibrations with fixed codomain C. It is established in the second section, by making use of an alternative construction of the join operation, that the homotopy fiber at x of the slice fibration $X/y \to x$ is the mapping space of maps from x to y in the ∞-category X, which is exploited in the third section to study final objects. In the fourth section one revisits Quillen's famous theorem A after Grothendieck, Joyal and Lurie, where the notions of proper functors and smooth functors are introduced. The sixth section addresses Quillen's theorem B after the highly technical fifth section on fully faithful and essentially surjective functors through the lenses of the covariant and contravariant model category structures.

Chapter 5 aims to construct the ∞-category of ∞-groupoids $\mathcal{S}$ with a smallness condition determined by a given universe to keep $\mathcal{S}$ small itself. The first section is a complement to §2.4, establishing that any fibration can be approximated by a minimal fibration, and that weak equivalences between minimal fibrations are always isomorphisms. The second section defines the universal left fibration with small fibers $p_{univ} : \mathcal{S}_{\cdot} \to \mathcal{S}$ and establishes (Theorem 5.2.14) that the homotopy theory of left fibrations over a simplicial set X is invariant under weak categorical equivalences. The third section is devoted to establishing that the correspondence between left fibrations over A and functors $A \to \mathcal{S}$ is homotopy-theoretic, involving a correspondence between the ∞-groupoid of invertible maps between two functors with values in $\mathcal{S}$ and the space of fiberwise equivalences between the associated left fibrations, which is extended in the fourth section to possibly non-invertible morphisms. The fifth section develops a homotopy theory of left bifibrations in the category of bisimplicial sets in order to define the Yoneda embedding, which is exploited in the sixth section (Propositions 5.6.2 and 5.6.5). The seventh section is devoted to comapring various versions of the notion of locally small ∞-category. The final section deals with the construction of the Yoneda embedding, establishing the so-called Yoneda lemma.

Chapter 6 develops adjoints, limits and Kan extensions. The first section interprets the derived functionality of the homotopy theory of left fibrations through the covariant model structures as adjunctions which are compatible with the Yoneda embedding in a suitable sense (Theorem 6.1,14), of which all the main constructions and features of this chapter are consequences. The first consequence is the various characterizations of adjoint pairs of functors claimed in Theorem 6.1.23. The second section addresses limits and colimits, showing that any presheaf on a small ∞-category is a canonical colimit of representable presheaves (Corollary 6.2.16), which is exploited in the third section to construct extensions of functors by colimits (Theorems 6.3.4 and Theorem 6.3.13). The fourth section studies Kan extensions as relative version of the notions of limit and colimit, particularly revisiting all the computations in §4.4 in terms of functors with values in $\mathcal{S}$. The fifth section deals with products, showing that they correspond in $\mathcal{S}$ to ordinary products of Kan complexes. The sixth section is concerned with the preceding one over an object, showing that all pullbacks are to be considered as binary products in sliced categories and demonstrating that for any right fibration $X \to A$ corresponding to a functor $F : A^{\mathrm{op}} \to \mathcal{S}$, presheaves on X correspond to presheaves on A over F. The final section revists the theme of the Yoneda embedding and of extensions of colimits in a relative way, leading to an equivalence of ∞-categories which interprets the operation $A \mapsto A^{\mathrm{op}}$ as a duality operator (Theorem 6.7.2).

Chapter 7 aims at providing tools to describe localizations of ∞-categories. The first section defines localizations through the appropriate universal properties, giving their general construction. The second section is concerned with calculus of fractions. The third section provides formulas to decompose limits as simple limits of smaller diagrams (Theorem 7.3.22). The fourth section investigates functors indexed by finite direct categories. The fifth section aims at explaining why the localization of any ∞-category with weak equivalences and fibrations has finite limts and constructing derived functors. The sixth section addresses necessary and sufficient conditions for a left exact functor to induce an equivalence of finitely complete ∞-categories. The seventh section shows how to ensure the existence of small products, hence of small limits, in the localization of an ∞-category with weak equivalences and fibrations. The eithth section, relying on all the preceding sections, establishes that the localization of the covariant model category over a simplicial set X is canonically equivalent to the ∞-category of functions from X to $\mathcal{S}$. The ninth section returnes to the problem of computing localizations of diagram categories. The tenth section explains how to compute mapping spaces. The final section gives a brief introduction to presentable ∞-categories, concluding with a result of Dugger which characterize presentable ∞-categories as localizations of combinatorial model structures.

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MSC:

18-02 Research exposition (monographs, survey articles) from category theory
18D05 Double categories, 2-categories, bicategories and generalizations (MSC2010)
18G55 Nonabelian homotopical algebra (MSC2010)

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